FUNCTIONAL ANALYSIS

PAPER- 402

1 mark questions

1. Define a metric space.
2. Give an example of a metric space.
3. Define an open set.
4. Give an example of an open set in a metric space.
5. Define a closed set.
6. Give an example of a closed set in a metric space.
7. Define a neighborhood in a metric space.
8. Give an example of a neighborhood in a metric space.
9. Define convergence in the context of sequences in a metric space.
10. Give an example of a convergent sequence in a metric space.
11. Define a Cauchy sequence.
12. Give an example of a Cauchy sequence in a metric space.
13. What is completeness in the context of metric spaces?
14. Define a continuous mapping between metric spaces.
15. Provide an example of a continuous function between two metric spaces.
16. Define a Banach space.
17. Give an example of a Banach space.
18. Define a normed space.
19. Give an example of a normed space.
20. What is a finite dimensional normed space?
21. Give an example of a finite dimensional normed space.
22. Define compactness in metric spaces.
23. Provide an example of a compact set in a metric space.
24. What is a finite dimensional linear operator?
25. Give an example of a finite dimensional linear operator.
26. Define bounded linear operators.
27. Provide an example of a bounded linear operator.
28. Define a linear functional.
29. Give an example of a linear functional on a finite dimensional space.
30. Define normed spaces of operators.
31. Provide an example of a normed space of operators.
32. Define an inner product space.
33. Give an example of an inner product space.
34. Define a Hilbert space.
35. Give an example of a Hilbert space.

36. Define an orthonormal set.
37. Provide an example of an orthonormal set.
38. What is a total orthonormal set?
39. Give an example of a total orthonormal set.
40. Explain the concept of representation of a functional on a Hilbert space.
41. Define self-adjoint operators.
42. Define unitary operators.
43. Define normal operators.
44. Give an example of a self-adjoint operator.
45. Give an example of a unitary operator.
46. Name one fundamental theorem for normed and Banach spaces.
47. What is Zorn's Lemma?
48. Explain the Hahn-Banach theorem.
49. What is meant by "application to bounded linear function on C" in the context of functional analysis?
50. Give an example of the application of the Hahn-Banach theorem to a bounded linear function on C.
51. A _______ is a set together with a function that assigns each pair of points in the set a non-negative real number that satisfies the triangle inequality.
52. The real numbers with the standard metric is an example of a _______.
53. A subset of a metric space is _______ if, for every point in the subset, there is a neighborhood around that point that is entirely contained in the subset.
54. The set of all real numbers less than 1 is an example of a _______ set in the real numbers.
55. A subset of a metric space is _______ if it contains all of its limit points.
56. The set of all real numbers less than or equal to 1 is an example of a _______ set in the real numbers.
57. A _______ of a point in a metric space is a set that contains an open set around the point.
58. The open interval (-1, 1) is a _______ of the point 0 in the real numbers.
59. A sequence in a metric space _______ to a limit if, given any positive distance, all but finitely many points of the sequence are within that distance of the limit.
60. The sequence 1/n _______ to 0 in the real numbers.
61. A sequence in a metric space is a _______ sequence if, given any positive distance, all terms of the sequence are eventually that close to each other.
62. Every sequence in a compact metric space has a _______ subsequence.

63. A metric space is _______ if every Cauchy sequence in the space converges to a limit in the space.
64. The real numbers with the standard metric is a _______ metric space.
65. A function between metric spaces is _______ if the preimage of every open set is open.
66. A _______ space is a complete normed vector space.
67. The space of continuous functions on a closed interval, with the sup norm, is an example of a _______ space.
68. A _______ space is a vector space equipped with a function that assigns a non-negative real number to each vector in the space, satisfying certain properties.
69. The space of sequences of real numbers that converge to 0, with the sup norm, is an example of a _______ space.
70. A normed space (or subspace) that has a finite basis is a _______ normed space (or subspace).
71. A subset of a metric space is _______ if every open cover of the subset has a finite subcover.
72. A _______ operator on a finite-dimensional vector space is a linear transformation between vector spaces.
73. A linear operator is _______ if there exists a constant C such that ||T(x)|| <= C||x|| for all x.
74. A _______ functional is a linear map from a vector space to its field of scalars.
75. The set of all _______ operators from one normed space to another, with the operator norm, is itself a normed space.
76. An _______ space is a vector space with an additional structure that allows you to compute the angle and length of vectors.
77. A _______ space is a complete inner product space.
78. A set of vectors is _______ if all vectors have a norm of 1 and are orthogonal to each other.
79. An _______ set is an orthonormal set that is dense in the space.
80. The _______ Representation Theorem states that every continuous linear functional on a Hilbert space can be represented as an inner product with a fixed vector in the space.
81. An operator on a Hilbert space is _______ if it is equal to its own adjoint.
82. An operator on a Hilbert space is _______ if it preserves the inner product.
83. An operator on a Hilbert space is _______ if it commutes with its adjoint.
84. The _______ Category Theorem is a fundamental result in the theory of Banach spaces.

85. The _______ Boundedness Principle is a fundamental result in the theory of Banach spaces.
86. The _______ Mapping Theorem is a fundamental result in the theory of Banach spaces.
87. The _______ Graph Theorem is a fundamental result in the theory of Banach spaces.
88. _______ Lemma is a principle of set theory stating that every partially ordered set in which every chain has an upper bound contains at least one maximal element.
89. The _______ Theorem states that every continuous linear functional on a normed vector space can be extended to the whole space.
90. In functional analysis, the _______ Theorem is frequently applied to bounded linear functions on C.

<u>2 marks questions</u>

1. Let (X, d) be a metric space and let A be a subset of X. Prove that a point x in X is a limit point of A if and only if every open ball B(x; r) around x for r > 0 intersects A at a point other than x.
2. Prove that the set of all bounded sequences in R, with the sup norm, forms a Banach space.
3. Let (X, ||.||) be a normed space and let T: X → X be a linear operator. Prove that T is bounded if and only if there exists a constant C > 0 such that ||T(x)|| ≤ C||x|| for all x in X.
4. Let K be a compact subset of a metric space (X, d). Prove that for any ε > 0, there exists a finite ε-net for K, i.e., a finite subset F of X such that for each x in K, there exists a point f in F with d(x, f) < ε.
5. Let (H, <., .>) be an inner product space, and let x, y be elements of H. Prove the Parallelogram Law: $||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$.
6. Let (H, <., .>) be a Hilbert space and let x, y, z be elements of H. Prove that <x+y, z> = <x, z> + <y, z> and <x, y+z> = <x, y> + <x, z>.
7. Prove the Baire Category Theorem for complete metric spaces: In a complete metric space, the intersection of countably many dense open subsets is dense.
8. Let (X, ||.||) be a normed space. Define a dual space X*. Prove that X* is also a normed space with the norm ||f|| = sup{ |f(x)| : x in X, ||x|| ≤ 1 }.
9. Prove that a linear operator T on a Hilbert space H is self-adjoint if and only if <Tx, y> = <x, Ty> for all x, y in H.
10. Let (X, ||.||) be a normed space and let T: X → X be a linear operator. Prove the Closed Graph Theorem: If the graph of T is closed in X x X, then T is bounded.

11. Define completeness in a metric space. Provide an example of a complete and a non-complete metric space.
12. What is the definition of a continuous function between two metric spaces? Provide an example.
13. Describe a Banach space. Why is every finite-dimensional normed space a Banach space?
14. What are the properties of a normed space? Give an example of a normed space that is not a subset of the real numbers.
15. Discuss the properties of finite-dimensional normed spaces and subspaces. Provide examples to illustrate.
16. Explain the concept of compactness in a metric space. Provide an example, demonstrating why the given set is compact.
17. Discuss the properties of a finite-dimensional linear operator. Provide an example of such an operator.
18. Discuss the concept of a dual space. Provide an example.
19. Discuss the concept of weak convergence. Provide an example to illustrate.
20. Discuss the concept of strong convergence. Provide an example to illustrate.

6/7 marks questions

1. Define a metric space. What properties does a function need to be a metric? Provide an example of a metric space that is not a subset of the real numbers.
2. Describe the concept of an open set in a metric space. Provide an example and show why it is open.
3. What is the definition of a closed set in a metric space? Provide an example and show why it is closed.
4. Define a neighborhood of a point in a metric space. Give an example demonstrating this concept.
5. Discuss the concept of convergence in a metric space. Give an example of a convergent sequence in a non-Euclidean metric space.
6. What is a Cauchy sequence? Provide an example in the space of rational numbers.
7. What is a bounded linear operator? Give an example and prove that it is indeed a bounded operator.
8. Define the concept of a linear functional. Provide an example of a linear functional on a finite-dimensional space.
9. Discuss the properties of normed spaces of operators. Provide an example of such a space.

10. Define an inner product space and its properties. Provide an example of an inner product space, demonstrating the properties.
11. Discuss the concept of a Hilbert space. Provide an example of a Hilbert space.
12. What is an orthonormal set in an inner product space? Give an example and prove that it is indeed orthonormal.
13. Define a total orthonormal set in a Hilbert space. Give an example and prove it is indeed a total orthonormal set.
14. Discuss the representation of functionals on a Hilbert space. Provide an example to illustrate.
15. What is the definition of a self-adjoint operator on a Hilbert space? Give an example.
16. What is a unitary operator on a Hilbert space? Give an example.
17. Discuss the concept of a normal operator on a Hilbert space. Give an example.
18. Discuss the fundamental theorems for normed and Banach spaces. Provide examples to illustrate each.
19. Explain Zorn's Lemma. Provide an example of its application.
20. Discuss the Hahn-Banach theorem. Provide an example to illustrate its application.
21. Describe how to apply the Hahn-Banach theorem to bounded linear functions on C. Provide an example.
22. Define the concept of an operator norm and discuss its properties. Give examples to illustrate.
23. Discuss the differences between a pre-Hilbert space and a Hilbert space. Provide examples to illustrate.
24. Discuss the concept of an adjoint operator. Provide an example and prove that it is indeed the adjoint.
25. Explain the Riesz Representation Theorem for Hilbert spaces. Provide an example to illustrate its application.
26. Discuss the projection theorem in a Hilbert space. Provide an example to illustrate its application.
27. Define a bounded operator and discuss its properties. Provide an example of a bounded operator.
28. Explain the concept of a compact operator. Provide an example.
29. Define an isometry in a metric space. Provide an example to illustrate.
30. Discuss the concept of a dense subset. Provide an example to illustrate.
31. Discuss the properties of a separable space. Provide examples of separable and non-separable spaces.
32. Define the spectrum of an operator. Discuss its properties.

33. Discuss the concept of a resolvent set. Provide an example to illustrate.
34. Discuss the properties of a self-adjoint operator. Provide an example.
35. Discuss the concept of a reflexive space. Provide an example.
36. Define a positive operator and discuss its properties. Provide an example.
37. Discuss the concept of a contraction mapping. Provide an example to illustrate.
38. Explain the Banach Fixed Point Theorem. Provide an example to illustrate its application.
39. Discuss the concept of an equivalent norm. Provide an example to illustrate.
40. Explain the Open Mapping Theorem. Provide an example to illustrate its application.