Core paper –XIII
COMPLEX ANALYSIS

I - Each question contain one mark

1 . If Z = $\frac{\sqrt{3}}{2}+\frac{i}{2}$ then find z^4=?

2 . Write the C.R equation in polar form .

3 . What is harmonic function ?

4 . At point where the function is not analytic is called ___________ .

5 . The Cauchy Integral formula ________ .

6 . The R.O.C of $\sum\frac{z^n}{n}$ is ?

7 . Write Morera's theorem .

8 . If C is circle | z|= 1 , then $\int zdz$ = ?

9 . L {z} = $\frac{e^z}{[z-1]\ 3}$ has a pole of order – at z=1

10. The function l [z] = $e^{\frac{1}{z}}$ has a isolated essential singularity at Z = ________ .

11 . If Z=1+$\sqrt{3}$ i. Find arg Z

12.If C is closed counter | z|= r & n ≠ -1 then $\oint z^n$ dz = ?

13. A function is analytical at every finite region , then the function is called _______ .

(a) a + b (b) Length of Arc C (c) $2\pi i$ (d) None

14. If $\emptyset(\mathrm{x}, \mathrm{y})$satisfy the Laplace equation then $\emptyset$is called __________

15. An analytic function f (z) = u +iv such that u & v satisfy Laplace Differential equation , then u & v are called __________________.

a. non – analytic b. harmonic c. analytic d. none

16. If f (z) is analytic function in in and on a closed contour C then $\oint f(z)$ dz = ?

17. If f (z) = e^z & C is unit circle |z|= 1 , then $\oint f(z)$ dz = ?

18. Write Cauchy Integral formula .

19. Write Morera's theorm .

20. Which of the following function is not analytic

a. sin z b. cos z c. az + b d. 1 / (z-a)

21. **An analytic function with constant modulus is ___________.**

22. **Real part of** $f(z) = \log z$ **is___________.**

23. **If n is a positive integer, then** $(1 + i\,3)\,n + (1 - i\,3)\,n$ **is equal to____________**

24. **Value of** $(1 - i)10 + (1 + i)10$ **equals __________**

25. **Real part of** $f(z) = z\,3$ **is ___________**

26. **If** $f(z) = (x^2 + ay^2 - 2xy) + i(bx^2 - y^2 + 2xy)$ **is analytic, then value of** a and b **is__________**

27. **If G is an open set in complex plane and** $f: G \to C$ **is differentiable, then on G,** f **is ___________**

28. **Value of** $(1 + i)24$ **is ____________**

29. **For any complex number** z **,** $e\,z \le 1$ **if**

A) Re $z \ge 0$ B) Re $z \le 0$
C) Im $z \le 0$ D) None of these

30. **For any complex number z ,** $e^{z+\pi i}$ **equals _________**

31. **For any real number** θ , $\frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$ = __________

32. **If S and T are domains in the complex plane , which of the following need** ***NOT*** **be true?**

A) $S \cup T$ is a domain if $S \cap T \neq \theta$ B) $S \cup T$ is an open set
C) $S \cap T$ is an open set D) $S \cap T$ is a domain

33. **The derivative of f(z) =** $\left(\frac{z^2-1}{z^2+1}\right)^{100}$ **at z ≠ ±i is _____________**

34. **If** $f(z)$ **is a real valued analytic function in a domain D, then :**

A) $f(z)$ is a constant B) $f(z)$ is identically zero

C) $f(z)$ has modulus 1 D) None of these

35. **If v is a harmonic conjugate for u , then a harmonic conjugate for v is _______**

36. **If f(z) is analytic and non zero in a domain D , then in D ,** $\ln f(z)$ **is**

____

37. **The function** e^{iz} **has period __________.**

38. **For real numbers** x **and** y **,** $\sin(x+iy)$ **equals :**

A) $SinxCoshy + i \cos xSinhy$ B) $CosxCoshy - iSinxSinhy$
C) $SinxCoshy - i \cos xSinhy$ D) $CosxCoshy + iSinxSinhy$

39. **The function Log z is analytic at:**
A) all points in the complex plane
B) all non zero complex numbers
C) all complex numbers except that on the non positive real axis
D) all complex numbers except that on the non positive imaginary axis

40. **Let G be a region not containing 0. Which of the following functions is *NOT* harmonic in G?**
A) $x+y$ B) x^2+y^2
C) x^2-y^2 D) $\log(x^2+y^2)$.

41. **Which of the following is not harmonic?**
A) $u = 2x(1-y)$ B) $u = 2xy + 3y^2 - 2y^3$
C) $u = 3x^2y + 2x^2 - y^3 - 2y^2$ D) **None of these**

42. **If** v **is the imaginary part of an analytic function** f **, an analytic function with real part** v **is**
given by:
A) $\frac{1}{f}$ B) $-f$
C) if D) $-if$

43. **Value of the limit** $\lim_{z\to 0}\frac{z}{\bar{z}}$ **equals _________**

44. **Real part of the function** $f(z)=|z|^2$ **equals _____________**

45. **For any complex number** z **,** $\exp(z+2\pi i)$ **equals __________**

46. If C is a circle $|z| = 1$ then $\int_c \bar{z}\, dz$ is ______________.

47. Which of the following function does the represent the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ For $|z| < \infty$

(a) Sin z **(b) cos z**

(c) e^z **(d) log(1+z)**

48. If a function is analytic at all points of a bounded domain except at finitely many points, then these exceptional points are called ______________.

49. A function which has poles as its only singularities in the finite part of the plane is said to be ___________.

50. If C is circle $|z - a| = r$, then $\int_C \frac{dz}{z-a}$ is ______________.

II – Each questions contains 2 or 3 marks .

1. What is the necessary condition for f(z) to be analytic give an example .
2. What is the sufficient condition for f(z) to be analytic give an example .
3. Find the R.O.C of $\sum\ (logn)^n z^n$
4. Find the R.O.C of $\Sigma z^n/(2^n$+1)
5. Write the Couchy's Residue Theorem .
6. The number of singular points of f (z) = $\frac{z+3}{z^2(z^2+2)}$
7. Evaluate $\int \frac{z-3}{z^2+2z+5}$ dz where C is a circle |z|=1
8. f(z) = $\frac{\sin z}{z^3}$ is a pole at z=0 of order = ?
9. If L= $\lim_{z \to i} \frac{iz^3-1}{z+i}$. then value of L= ?
10. If $\lim\limits_{z \to 2i} \frac{z^2+4}{z-2i}$ = α then value of α = ?
11. Find the pole of f(z) = $\frac{1}{z(z+1)^2}$?
12. The pole of f(z) = $\frac{z^2+1}{(z-1)(z+1)}$ at z=1 ?
13. f(z) = $\frac{z^2+z+1}{(z-1)(z-3)}$ then $\int_C f(z)dz$ = ? , where c is |z| = $\frac{1}{2}$.
14. show that u=$\frac{1}{2}\log x^2 + y^2$ is harmonic.
15. Find the domain of convergence of the power series $\sum \left(\frac{2i}{z+i+1}\right)^n$.

16.Evaluate $\int \frac{e^z}{z-2}dz$ where c is a circle |z|=3 .

17.Write Morera's theorem .

18.The residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at z=1,2,3 are respectively __ , __, __ .

19.Residue of the function $\frac{1}{(z^2+1)^3}$ at z=i is ?

20. **An arc** $z = z(t)$; $a \le t \le b$ **is simple if :**

A) $z(t)$ is continuous B) $z(t)$ is a one to one function

C) $z(t)$ is such that $z(a) = z(b)$ D) None of these

21.**Which of the following is not a simply connected region?**

A) circular disk B) half planes

C) an annulus region D) a parallel strip

22.**Which of the following subset of C is a simply connected region?**

A) $\{z \mid ;0 < |z| < 1\}$ B) $\{z \mid ;0 < |z| \le 4\}$

C) $\{z \mid ;1 < |z| < 2\}$ D) $\{z \mid ;0 \le |z| < 3\}$

23.**Let τ be any circle enclosing the origin and oriented counter clockwise. Then the value of the integral** $\int_\tau \frac{\cos z}{z^2}dz$ **is** ___________

24.**The integral** $\int_{|z|\to 2\pi} \frac{\sin z}{(z-\pi)^2}dz$ **where the curve is taken anti-clockwise, equals** ___________

25.**The value of the integral** $\int_c \frac{dz}{(z-a)^{10}}$ **, where C is |z - a| = 3 is** ________

26.**The only bounded entire functions are:**

A) Real valued functions B) harmonic functions

C) Constant functions D) Exponential function

27.**Suppose** $f(z)$ **is analytic inside and on unit circle. If** $f(z) \le 1$, $\forall z$ with $z = 1$ **. Then an upper bound for** $|f^n(0)|$ **is** _____________

28.The integral $\oint_c \frac{dz}{z^2-1}$ **around a closed curve containing -1 but not 1, has the value**___________

29.**The value of the integral** $\int_c \frac{e^{5z}}{z^3}dz$ **, where C is |z | = 3 is** ____________.

30.**Value of the integral** $\int_0^\pi e^{it}dt$ **is** _________.

31.**Value of** $\oint_c \frac{3z-2}{z^2-z}dz$ **where C is** $z = 2$ **is** ___________.

32.**If n is any non zero integer, then** $\int_0^{2\pi} e^{in\theta}d\theta$ **equals** __________.

33. **The value of the integral $\int |z| dz$ evaluated and the semi circle** $|z| = 1$; $0 \le \arg z \le \pi$ *starting at z=1 is.*__________

34. **Converse of Cauchy's integral theorem is known as_____________.**

35. **The value of the integral $\frac{1}{2\pi i}\int_{|z|=1} \frac{\cos z}{z^3} dz$ is _____________.**

36. **The parametric equation of the semi circle of radius 1 with center at 0, lying in the upper half plane from the point 1 to -1 is:**

A) $z(t) = e^{it}$; $0 \le t \le \pi$ B) $z(t) = e^{it}$; $-\pi \le t \le \pi$

C) $z(t) = e^{2it}$; $0 \le t \le \pi$ D) $z(t) = e^{it}$; $0 \le t \le 2\pi$

37. The integral $\int_C \frac{ze^z}{z^2+9} dz$ **has non zero value if** C **is :**

A) $|z| = 1$ B) $|z| = 2$

C) $|z-1| = 1$ D) $|z| = 4$

38. **If** f **is continuous in a domain D and if $\int_C f(z)dz = 0$ for every simple closed positively oriented contour** C **in D, then:**

A) f is a constant in D B) f is analytic in D

C) f is real valued in D D) f is purely imaginary in D

39. **If** f **is a analytic within and on a simple closed, positively oriented contour** C **and if Z_0 is a point interior to** C**, then $\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$ equals _______________**

40. **The radius of convergence of the power series of the function f(z) = $\frac{1}{1-z}$ about z=$\frac{1}{4}$ is ___________.**

41. The coefficient of 1/z in the Laurent series expansion of f(z) = $\frac{z}{z(z-2)}$ in the region 2<$|z|$<∞ is _____

42. If f(z) is entire , then f(z) = $\sum_{n=1}^{\infty} a_n z^n$ has radius of convergence ________

43. If f(z) = $\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ is represented as the Laurent series, then z_0 is a removable singularity of f(z) if

A)$a_n = 0$, for n > 0 B) $a_n = 0$, for n < 0

C) $a_n = 0$, for n ≥ 0 D) $a_n = 0$, for n ≤ 0

44. **A function** $f(z)$ **given by a power series is analytic at :**

A) Every point of its domain

B) every point inside its circle of convergence

C) every point on its circle of convergence

D) every point in the complex plane

45. **The singular points of the function f(z) = $\frac{1}{4z-z^2}$ are**

46. **The singular points of the function f(z) = $\frac{e^z}{z(z^2+1)}$ that lies inside $|z-i| = \frac{3}{2}$ are :**

A) $z = 0$ and $z = -i$ B) $z = 0$ and $z = i$

C) $z = i$ and $z = -i$ D) $z = 0$ and $z = 1$

47. **The power series $b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots\ldots$ Converges :**

A) inside of some circle | z |=R B) on the circle | z |= 1

C) on some circle | z |=R D) outside of some circle | z |=R

48. Number of poles of the function f(z) = $\tan\frac{1}{z}$ is ________.

49. Number of zeros of the function f(z) = $\sin\frac{1}{z}$ is __________.

50. The number of isolated singular points of f(z) = $\frac{z+3}{z^2(z^2+2)}$ is _____________.

All the question contain 8 marks :

1. Prove that the following functions are harmonic and find its harmonic conjugate : $e^{-x}(x\cos y + y\sin y)$.
2. If f(z) is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re(z)|^2 = 2|f'(z)|^2$.
3. Prove that $|z|^2$ is continuous everywhere but nowhere differentiable except at origin .
4. If w = $\log z$, find $\frac{dw}{dz}$ and determine where w is non analytic.
5. Show that the function f(z)=u+iv where f(z) = $\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$, z ≠ 0 and f(0)=0 is continuous and that CR equations are satisfied at origin, yet f'(z) does not exist.
6. Determine the pole of f(z) = $\frac{z^2}{(z-1)^2+(z+2)}$ and residues at each point . Hence evaluate $\int_c f(z)dz$ where c is circle $|z|$ = 2.5
7. Evaluate the residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at z=1,2,3 and ∞ and show that their sum is zero.
8. State and prove residue theorem.
9. Write argument principle . Evaluate $\int \frac{f'(z)}{f(z)}dz$ where f(z)=$\frac{(z^2+1)^2}{(z^2+3z+2)^3}$ and c is circle |z|=3 taken in positive sense.

10. $\int_\gamma \frac{z}{(z^2+1)(z-3)^2}dz$, γ = {z : |z|=2} positively oriented.

11. f(z) = $\frac{z^2+16}{(z-i)^2(z+i)}$, find the singularities also find the residue at that point.

12. State and prove Cauchy Integral Formula.

13. Find the modulus and argument of $\frac{2+i}{4i+(1+i)^2}$ and also convert it into polar form.

14. Find the Radius of convergence of $\sum \frac{n\sqrt{2}+i}{1+i2n} z^n$.

15. Find the region of convergence of the series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$.

16. If u = $\frac{\sin 2x}{\cosh 2y + \cos 2x}$, find the corresponding analytic function f(z) = u+iv.

17. Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$.

18. If u-v = (x-y)(x^2+4xy+y^2) and f(z)=u+iv is an analytic function of z=x+iy, find f(z) in terms of z.

19. If w=u+iv represents the complex potential for an electric field and v =x^2-y^2+$\frac{x}{x^2+y^2}$, determine the function u.

20. If f(z)=u+iv is an analytic function of z and u-v = $\frac{\cos x+\sin x-e^{-y}}{2\cos x-e^{y}-e^{-y}}$, find f(z) subject to the condition f($\frac{\pi}{2}$)=0.

21. If f(z)=u+iv is an analytic function of z=x+iy and u-v = $\frac{e^{y}-\cos x+\sin x}{\cosh y-\cos x}$, find f(z) subject to the condition f($\frac{\pi}{2}$) = $\frac{3-i}{2}$.

22. Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)}$dz where C:$|z-i|$=2.

23. Evaluate $\int_c \frac{z-3}{z^2+2z+5}$dz where C is circle i) $|z| = 1$, ii) $|z+1-i| = 2$.

24. Evaluate the following integrals by using Cauchy's integral formula

$\int_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)} dz \quad where\ C\ represents\ the\ circle\ |z| = 3.$

25. Evaluate the following integrals by using Cauchy's integral formula $\frac{1}{2\pi i}\int_c \frac{e^{zt}}{z^2+1} dz$,t>0 $where\ C\ represents\ the\ circle\ |z| = 3.$

26. Evaluate $\int_C \frac{e^z}{z-2} dz$ where C is the circle (a) $|z|$=1 and (b) $|z| = 3$.

27. Evaluate $\int_C \frac{z^2-4}{z(z^2+9)} dz$ where C is the circle $|z| = 1$.

28. Evaluate $\int \frac{e^{ax}}{z^2+1} dz$ where C is the circle $|z| = 2$.

29. Evaluate by Cauchy's integral formula $\int_C \frac{dz}{z(z+\pi i)}$, where C is $|z+3i|$ = 1.

30. Evaluate $\int_C \frac{e^{3z}}{z+i}$ if C is the circle $|z+1-i| = 2$.

31.Find the value of $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ if C is the circle $|z| = 1$.

32.Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where the path of integration C is $|z| = 3$.

33. Using Cauchy integral formula, calculate the following integral $\int_c \frac{z\, dz}{(9-z^2)(z+i)}$, where C is the circle $|z| = 2$ described in positive sense.

34. Using Cauchy integral formula, calculate the following integral $\int_c \frac{\cosh \pi z\, dz}{(z^2+1)z}$, where C is the circle $|z| = 2$.

35. Using Cauchy integral formula, calculate the following integral $\int_C \frac{e^{az}\, dz}{(z-\pi i)}$, where C is the ellipse $|z-2| + |z+2| = 6$.

36. Using Cauchy integral formula, calculate the following integral $\int_C \frac{dz}{z-2}$, where C is $|z| = 3$.

37.Evaluate $\int_C \frac{\tan\left(\frac{z}{2}\right)}{(z-x_0)^2} dz$, where C is the boundary of the square whose sides lie along the lines x = ±2 , y = ±2 and it is described in positive sense, where $|x_0| < 2$.

38.Evaluate $\int_c \frac{dz}{z^2+2z+2}$ where C is the square having vertices at (0,0),(-2,0),(-2,-2),(0,-2) oriented in anticlockwise direction .

39. If C is the unit circle about the origin , described in positive sense , show that

(a) $\int_C \frac{e^{-z}}{z^2} dz = -2\pi i$ and (b)$\int_C \left(\frac{\sin z}{z}\right) dz = 0$.

40.Evaluate $\int_C \frac{\sin z}{\left(z-\frac{\pi}{4}\right)^3} dz$ where C is $\left|z-\frac{\pi}{4}\right| = \frac{1}{2}$.

41. Show that $\int_0^{2\pi} \frac{d\theta}{a+b\cos 2\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{(a^2-b^2)}}$.

42. Show that $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{3}$, $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}$.

43. Show that $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{(1-a^2)}}$, $a^2 < 1$.

44. Prove that $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{(a^2-b^2)}}$.

45. Prove that $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$.

46.Use the method of contour integration to prove that $\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \frac{2\pi}{1-a^2}$,0<a<1.

47. Evaluate $\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta} d\theta$, a>1 .

48. Apply the method of contour integration to prove that $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \frac{\pi}{6}$.

49. By the method of contour integration prove that $\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta = \frac{\pi a^2}{1-a^2}$, (-1<a<1).

50. Prove that $\int_0^{2\pi} \frac{sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left\{a - \sqrt{(a^2 - b^2)}\right\}$.